Answer of Final Exam

Geometry May 2014 For preparatory year

Question (3)

(a) Write down the equation for a rotation of axis through an angle $\pi/4$. Hence prove that the curve $2xy = a^2$ cane be transformed to $x^2 - y^2 = a^2$.

Answer

$x = \frac{1}{\sqrt{2}}(x - y)$, $\quad = \frac{1}{\sqrt{2}}(x + y)$.

$$2xy = 2\frac{1}{\sqrt{2}}(x - y)\frac{1}{\sqrt{2}}(x + y) = x^2 - y^2 = a^2$$

--- (5 marks)

(b) Find the equation of the parabola with focus at $(3, -4)$ and the directrix is $6x - 7y + 5 = 0$.

Answer

Let the point $(x,\ y)$ on the curve then the required equation is

$(x - 3)^2 + (y + 4)^2 = \left(\frac{6x-7y+5}{\sqrt{36+49}}\right)^2$

--- (8 marks)

(c) Find the equation of the ellipse whose foci $(\pm 4,\ 0)$ and its eccentricity is $1/3$.

Answer

$ae = a\frac{1}{3} = 4\ \ then \quad a = 12$

And $b^2 = a^2(1 - e^2) = 144\left(1 - \frac{1}{9}\right) = 128$

The required equation is $\quad \frac{x^2}{144} + \frac{y^2}{128} = 1$

--- (8 marks)

(d) Find the equation of the circle with center at $(6, 6)$ and touch the circle $x^2 + y^2 = 32$.

Answer

Since the circles are touch each other then the sum of two radii equal the distance between the two centers $= 6\sqrt{2}$

And the length of radius of $x^2 + y^2 = 32$. Is $4\sqrt{2}$ then the length of the radius of the required circle is $2\sqrt{2}$ and its equation is $(x - 6)^2 + (y - 6)^2 = 8$

$$x^2 + y^2 - 12x - 12y + 64 = 0$$

--- (5 marks)

(Total 26 marks)

Question (4)

(a) Find the equation of the two tangents of the hyperbola $9x^2 - 4y^2 = 36$ drawn from the point $(0, 9)$. Find the angle between them.

Answer

The line $y = mx + \sqrt{a^2m^2 - b^2}$ is a tangent for the hyperbola

$9x^2 - 4y^2 = 36$ or $\frac{9x^2}{36} - \frac{4y^2}{36} = \frac{x^2}{4} - \frac{y^2}{9} = 1$

We find that $a^2 = 4 and\ b^2 = 9$

If this line passes through the point $(0, 9)$. Then $9 = \sqrt{a^2m^2 - b^2}$

Substitute about a and b from equation of the curve then

$$9 = \sqrt{4m^2 - 9}$$

$$90 = 4m^2$$

$$m = \pm\frac{3\sqrt{10}}{2}$$

The two tangent are

$2y - 3\sqrt{10}x - 18 = 0$ and $2y + 3\sqrt{10}x - 18 = 0$

$$tan\theta = \pm$$

$$\frac{m_1 - m_2}{1 + m_1m_2} = \pm\frac{3\sqrt{10}/2 - (-3\sqrt{10})/2}{1 + (3\sqrt{10}/2)(-3\sqrt{10}/2)} = \pm\frac{3\sqrt{10}}{1 - 22.5} = \pm\frac{3\sqrt{10}}{-21.5} = 0.4412480$$

Another solution

$(9x^2 - 4y^2 - 36)(9{x_1}^2 - 4{y_1}^2 - 36) = (9xx_1 - 4yy_1 - 36)^2$

Substitute by the point $(0, 9)$ then

$(9x^2 - 4y^2 - 36)(-4(9)^2 - 36) = (-4y(9) - 36)^2$

$(9x^2 - 4y^2 - 36)(-360) = (-36y - 36)^2$

$(9x^2 - 4y^2 - 36)(-360) = (-36)^2(y + 1)^2$

$(9x^2 - 4y^2 - 36)(-10) = 36(y + 1)^2$

$(9x^2 - 4y^2 - 36)(-5) = 18(y^2 + 2y + 1)$

$(45x^2 - 20y^2 - 180) = (-18y^2 - 36y - 18)$

$(45x^2 - 2y^2 + 36y - 162) = 0$

$tan\theta = \pm\frac{2\sqrt{h^2-ab}}{a+b} = \pm\frac{2\sqrt{-(45)(-2)}}{43} = 0.441248$

--- (8 marks)

(b) Prove that the circle $x^2 + y^2 - 2ax - 2ay + a^2 = 0$ touch the axis.
Hence find the equation of the circle which touches the axis at a distance 4 from the origin.

Answer

$$x^2 + y^2 - 2ax - 2ay + a^2 = 0$$

Cane be written as

$$(x - a)^2 + (y - a)^2 = a^2$$

Which show that the center at (a, a) and the radius equal a. and the circle touch the axis

Another solution

put $x = 0$ in the equation

$$x^2 + y^2 - 2ax - 2ay + a^2 = 0$$

then $y^2 - 2ay + a^2 = 0$ and $(y - a)^2 = 0$

$$y = a$$

The intercept of $y - axis$ one point then the circle touch $y - axis$ The intercept of $x - axis$ one point then the circle touch $x - axis$

The required equation is the same with $a = \pm 4$

$$x^2 + y^2 - 2(\pm 4)x - 2(\pm 4)y + (\pm 4)^2 = 0$$

$$x^2 + y^2 \pm 8x \pm 8y + 16 = 0$$

--- (8 marks)

(c) What conic the equation $4x^2 - 9y^2 - 16x + 54y - 101 = 0$ represent ? find its foci and equation of its directrix.

Answer

The equation cane be written as

$$(4x^2 - 16x + 16) - (9y^2 - 54y + 36) - 20 - 101 = 0$$

$$(2x-4)^2-(3y-6)^2=36$$

$$4(x-2)^2-9(y-2)^2=36$$

$$\frac{4(x-2)^2}{36}-\frac{9(y-2)^2}{36}=1$$

$$\frac{(x-2)^2}{9}-\frac{(y-2)^2}{4}=1$$

Which is a horizontal hyperbola with center at $(\alpha,\beta)=(2,3)$ and

$b^2=a^2(e^2-1)$ and $e^2=\frac{13}{2}$ $\quad\rightarrow\quad$ $e=\frac{\sqrt{13}}{3}$

Foci at $(\alpha \pm ae,\beta)=(2\pm\sqrt{13},\ 2)$ Directrix are $x=\alpha\pm\frac{a}{e}=2\pm\frac{9}{\sqrt{13}}$

Note that

We can find the center by solving the two equations

$8x-16=0$

$-18y+54y=0$

Which show that the center is $(\alpha,\beta)=(2,3)$

-- (8 marks)

(Total 24 marks)